曲线积分与曲面积分

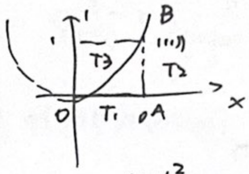
第一类: $\int_\Gamma f(x,y,z)ds = \lim_{\|T\|\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta s_i \Rightarrow \int_\alpha^\beta f(x(t),y(t))\sqrt{x'^2(t)+y'^2(t)}\,dt$

极: $C: r=r(\theta)\ \alpha\le\theta\le\beta$ 则 $\int_C f(x,y)ds = \int_\alpha^\beta f(r(\theta)\cos\theta, r(\theta)\sin\theta)\sqrt{r^2(\theta)+(r'(\theta))^2}\,d\theta.$

(1): $\int_C (x+2y-z)ds = \int_0^1 (1-t+2-4t-1-2t)\sqrt{1+4+4}\,dt$ $C:\overline{AB}$: $A(1,1,1)\ B(0,-1,3)$ $\begin{cases}x=1-t\\ y=1-2t\\ z=1+2t\end{cases}$ $t\in[0,1]$

$= 3\int_0^1 (2-7t)dt = 3\cdot(2t-\frac{7}{2}t^2)\Big|_0^1 = 3(2-7/2) = -\frac{9}{2}$

(2): $\oint_C x ds = \int_{OA} x ds + \int_{AB} x ds + \int_{BC} x ds$ $C: \begin{cases}y=x^2\\ x=1\\ y=0\end{cases}$



$= \int_0^1 x dx + \int_{\cancel{AB}0}^1 1 ds + \underline{\int_0^1 t\sqrt{1+4t^2}\,dt}$ $\begin{cases}y=t^2\\ x=t.\end{cases}$

$= \frac{1}{2}x^2\Big|_0^1 + 1 + \frac{1}{8}\int_0^1 \sqrt{1+4t^2}\,d(4t^2+1) = \frac{1}{2}+1+\frac{1}{8}\cdot\frac{2}{3}(\sqrt{4t^2+1})^3\Big|_0^1$

$= \cancel{x}\frac{3}{2} + \frac{1}{12}(\sqrt{5}^3-1) = 3/2 + \frac{1}{12}(5\sqrt{5}-1) = \frac{5\sqrt{5}}{12} + 17/12$

(3): $\int_L \sqrt{4a^2-x^2-y^2}\,ds$: $L: x^2+y^2=2ax$: $(x-a)^2+y^2=a^2$

$\begin{cases}x=a+a\cos\theta\\ y=a\sin\theta\end{cases}$ $\Rightarrow ds=\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}\,d\theta = a\,d\theta$

$= \int_0^{2\pi}\sqrt{4a^2-2ax}\ \cancel{a}d\theta$: $\int_0^{2\pi}\sqrt{4a^2-2a(a+a\cos\theta)}\,a\,d\theta$

$= \int_0^{2\pi}\sqrt{2a^2-2a^2\cos\theta}\ a\,d\theta = \int_0^{2\pi} a^2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta = 2a^2\int_0^{2\pi}\sin\frac{\theta}{2}d\theta = 8a^2.$

$L: (x^2+y^2+z^2=a^2)\cap(x+y+z=0)$

(4): $\oint_L x^2 ds = \frac{1}{3}\oint_L (x^2+y^2+z^2)ds = \frac{a^2}{3}\oint_L ds = \frac{2}{3}\pi a^3$

$\oint_L xy ds = \frac{1}{3}\frac{1}{2}\oint_L ((x+y+z)^2-(x^2+y^2+z^2))ds = -\frac{1}{6}a^2\oint_L ds = -\frac{1}{3}\pi a^3$

$\oint_L x ds = \frac{1}{3}\oint (x+y+z)ds = \frac{1}{3}\oint 0\cdot ds = 0$

$\oint_L (2x-3y+4z)^2 ds = \oint_L (4x^2+9y^2+16z^2)ds + \oint_L(-12xy+16xz-24yz)ds$

$= 29\oint_L z^2 ds - 20\oint_L \boxed{ds}\,xy = 26\pi a^3$

$ds = \sqrt{1+y'^2}\,dx = \sqrt{1+\sin x}\,dx = (\sin\frac{x}{2}+\cos\frac{x}{2})\,dx.$

(5): $C: y=\int_0^x\sqrt{\sin t}\,dt$ $\int_C x ds = \int_0^\pi (\sin\frac{x}{2}+\cos\frac{x}{2})x\,dx \overset{\frac{x}{2}=u}{=\!=} 4\int_0^{\frac{\pi}{2}} u(\sin u+\cos u)\,du = 4[u(\sin u-\cos u)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(\sin u-\cos u)\,du]$

$=2\pi$

内：第2类曲线积分：

$$\int_C (P(x,y)dx+Q(x,y)dy) = \int_C Pdx+Qdy = \lim_{\lambda\to 0}\sum_{i=1}^{n}[P(\xi_i,\eta_i)\Delta x_i+Q(\xi_i,\eta_i)\Delta y_i]$$

$d\vec{s}=(dx, dy)$ 则：$\int \vec{F}\cdot d\vec{s}$. $F(x,y)=P(x,y)\vec{i}+Q(x,y)\vec{j}$

(6)：$I_R=\oint_{x^2+y^2=R^2}\frac{ydx-xdy}{(x^2+xy+y^2)^2}$ 证明：$\lim_{R\to\infty} I_R=0$

证明：$P(x,y)=\frac{y}{(x^2+xy+y^2)^2}$ $Q(x,y)=-\frac{x}{(x^2+xy+y^2)^2}$ $(x^2+y^2+xy)\geq\frac{1}{2}(x^2+y^2)$

则 $P^2+Q^2=\frac{(x^2+y^2)}{(x^2+xy+y^2)^4}\leq\frac{4}{(x^2+y^2)^3}$

再有：$|I_R|=|\oint_{x^2+y^2=R^2}(P\cos\alpha+Q\overset{\cos\beta}{\cancel{\sin\alpha}})ds|\leq\oint_{x^2+y^2=R^2}\sqrt{P^2+Q^2}ds\leq\oint_{x^2+y^2=R^2}\frac{2}{(x^2+y^2)^{3/2}}ds=\frac{4\pi R}{R^3}=\frac{4\pi}{R^2}$. 则证

算：$\int_C Pdx+Qdy=\underline{\int_\alpha^\beta[P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)]dt}$

$C: y=y(x)$: 则：$\int_C P(x,y)dx=\int_a^b P(x,y(x))dx$ $\int_C Q(x,y)dy=\int_a^b Q(x,y(x))y'(x)dx$

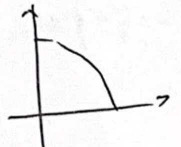
例：$C: x^2+y^2=a^2$. $\oint_C\frac{(x+y)dx-(x-y)dy}{(x^2+y^2)}=\int_0^{2\pi}\frac{(a\cos\theta+a\sin\theta)(-a\sin\theta)d\theta-(a\cos\theta-a\sin\theta)(a\cos\theta)d\theta}{a^2}$

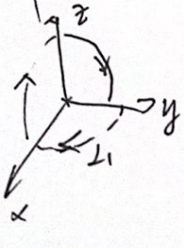
$\begin{cases}x=a\cos\theta\\ y=a\sin\theta\end{cases}\Rightarrow$ $=\int_0^{2\pi}\frac{(-a^2\sin^2\theta-a^2\cos^2\theta)d\theta}{a^2}=\frac{-a^2}{a^2}\int_0^{2\pi}d\theta=-2\pi$

(8)：$S:\{(x,y,z)\mid x^2+y^2+z^2=1,\ x,y,z\geq 0\}$ 逆时针. $L_1\subseteq\overset{z=0}{\underline{\underline{xoy}}}$. $L_2\subseteq\overset{x=0}{yoz}$. $L_3\subseteq\underline{zox}$.

则 $I=\int_L(y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz=(\int_{L_1}+\int_{L_2}+\int_{L_3})\vec{f}\cdot d\vec{s}$

$$=\underline{\int_{L_1}y^2dx-x^2dy}+\int_{L_2}z^2dy-y^2dz+\int_{L_3}x^2dz-z^2dx$$





$x^2+y^2=1$

$=\int_0^{\pi/2}\underline{(\sin^2\theta\cos\theta-\cos^2\theta(-\sin\theta))d\theta}+\cdots$

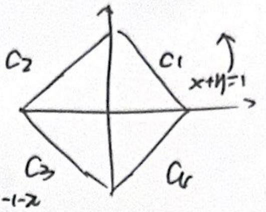
$=\int_0^{\pi/2}\sin\theta\cos\theta(\sin\theta+\cos\theta)d\theta+\cdots$

$=\int_0^{\pi/2}\sin^2\theta\, d\overset{\sin\theta}{\cancel{\cos\theta}}-\int_0^{\pi/2}\overset{\cancel{\cos^2\theta d\cos}}{\cos^2\theta d\cos\theta}+\cdots=\frac{1}{3}\sin^3\theta|_0^{\pi/2}-\frac{1}{3}\cos^3\theta|_0^{\pi/2}=$

$=\int_{\pi/2}^0(\sin^2\theta(-\overset{\sin\theta}{\cancel{\cos\theta}})d\theta-\cos^2\theta\cos\theta d\theta)$

$=\int_0^{\pi/2}\sin^3\theta d\theta+\int_0^{\pi/2}\cos^3\theta d\theta=4/3.$

(9): $\oint_C \frac{(x+y)dx-(x-y)dy}{(x^2+|x+y|)^2(y^2)} = I$ C: $|x|+|y|=1$



① $I = \oint_C \frac{(x+y)dx-(x-y)dy}{(|x|+|y|)^3} = \oint_C (x+y)dx-(x-y)dy = \iint_D (-1-1)dxdy = -4$

(Green)

② $I = \oint_C (x+y)dx-(x-y)dy = (\sum\int_{C_i})(x+y)dx-(x-y)dy = \int_1^0 (1-(2x-1)(-1))dx + \int_0^{-1} [(2x+1)+1)dx$

$+\int_{-1}^0 [-1-(2x+1)(-1))dx + \int_0^1 [(2x-1)-1)dx = -4$

Green 公式: P(x,y), Q(x,y) 在有界闭区域 ^{D} 上连续, 且有一阶偏导数, 则有:

$$\iint_D \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{vmatrix} dxdy = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \oint_{\partial D} Px+Qdy$$

注: 沿边界走 向区域在左边

方向为正向: 外边界逆时针, 内边界顺时针.

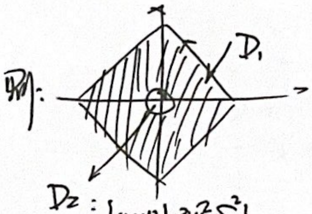
(10). $I_k = \oint_{C_k} \frac{xdy-ydx}{x^2+y^2} = \frac{1}{a^2}\oint_{C_k} xdy-ydx$ $\Bigg\{$ $C_k: x^2+y^2=a^2 \Rightarrow \begin{cases} x=a\cos\theta \\ y=a\sin\theta \end{cases}$ ~~否~~:

~~$=\frac{1}{a^2}\int a^2\cos\theta+\sin\theta$ 不对~~ $= \frac{1}{a^2}\iint_D$ ~~1+~~ $(1+1)dxdy = +\frac{2}{a^2}\cdot\pi a^2 = +2\pi$

II C_2: ~~★~~ $|x|+|y|=1$ (逆时针) $P=-\frac{y}{x^2+y^2}$ $Q=+\frac{x}{x^2+y^2}$ $\Rightarrow \frac{\partial P}{\partial y} = \frac{(-1)(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ "

则:



$D_2: \{(x,y)|x^2+y^2\le\delta^2\}$

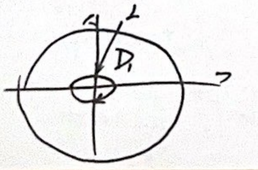
$\oint_{C_2} Pdx+Qdy = (\underset{\sim\sim}{\oint_{C_2+\partial D_2}} - \oint_{\partial D_2})Pdx+Qdy$

$\begin{cases} x=\delta\cos\theta \\ y=\delta\sin\theta \end{cases}$

$= \oint_{\underline{C_2+\partial D_2负向}} Pdx+Qdy - \oint_{\partial D_2负向} Pdx+Qdy =$ ~~0~~ $- \oint_{D_2负向} \frac{xdy-ydx}{\delta^2}$

$= +\frac{1}{\delta^2}\int_0^{2\pi} \delta^2\cos^2\theta - \delta^2\sin\theta(-\sin\theta)d\theta = 2\pi.$ ← $\oint_{D_2^-} \frac{xdy-ydx}{\delta^2}$ 变为逆时针.

(11): $C: x^2+y^2=1.$ $I = \oint_C \frac{xdy-ydx}{2x^2+y^2} = (\oint_{C+L} - \oint_L)\frac{xdy-ydx}{2x^2+y^2} = \oint_{L^-逆} \frac{xdy-ydx}{2x^2+y^2} = \frac{1}{\delta^2}\oint_L xdy-ydx = \frac{1}{\delta}\int_0^{2\pi} \frac{1}{\sqrt2}\delta^2\cos^2\theta - \delta\sin\theta(\delta(-\sin\theta))d\theta$

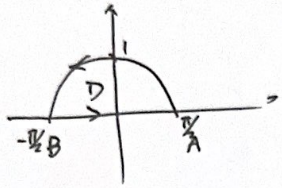


$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$x=\frac{1}{\sqrt2}\delta\cos\theta$ $y=\delta\sin\theta$

$=\sqrt2\pi.$

(12): $C: A(\frac{\pi}{2},0)\ B(0,\frac{\pi}{2},0)$. $y=\cos x$. $\int_C (e^x\sin y - x y)dx + (e^x\cos y + x^4)dy$.



$$= \oint_{C+\overline{AB}A} (e^x\sin y - x y)dx + (e^x\cos y + x^4)dy + \underline{\int_{\overline{AB}} (e^x\cos\sin y - x y)dx + (e^x\cos y + x^4)dy}$$

$x: \frac{\pi}{2} \to -\frac{\pi}{2}$, $y: 0$

~~1~~

$$= \iint_D ((-e^x\cos y + 4)\mp 4(e^x\cos y + \underset{=}{4x^3})dxdy + \quad 0$$

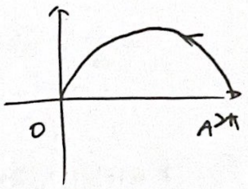
$$= \iint_D ((4x^3 + 4)dxdy \quad \cancel{\ast} = 4\iint_D dxdy = 8\int_0^{\frac{\pi}{2}} \cos x dx = 8\sin x\Big|_0^{\pi/2} = 8$$

~~$= 4\iint_D dxdy +$~~

(13): Green: $\oint_{\partial D} -y dx + x dy = \iint_D (1+1) dxdy = 2S(D) \Rightarrow \text{PS} = \frac{1}{2}\oint_{\partial D} -y dx + x dy$

$\Rightarrow$ For: $C_1 = x^2 + y^2 = a^2$: $S = \frac{1}{2}\int_0^{2\pi} (-a\sin\theta(-a\cos\theta) + a^2\cos^2\theta)d\theta = \frac{1}{2}a^2\cdot\theta\Big|_0^{2\pi} = a^2\pi$

C_2: $\begin{cases} x = a(t - \sin t) \\ y = a(1-\cos t) \end{cases}$ $S = \frac{1}{2}\oint_{C+\overline{OA}} -y dx + x dy = \frac{1}{2}\int_C -y dx + x dy + \frac{1}{2}\underline{\int_{\overline{AO}} -y dx + x dy}$ $\overset{0}{\shortparallel}$



$$= \frac{1}{2}\int_{2\pi}^{0} (-a(1-\cos t)a(1-\cos t) + a(t-\sin t)\overset{a\sin t}{\cancel{a(1+\sin t)}})dt$$

$$\boxed{= \frac{1}{2}\int_{2\pi}^{0} (\underline{a^2}(1-\sin^2 t) - a^2(1+\cos^2 t - 2\cos t))dt = \frac{1}{2}\int_{2\pi}^{0} \cancel{-2a} - a^2 + 2a^2\cos t\, dt}$$

$$\boxed{= \frac{1}{2}\int_0^{2\pi} a^2 - 2a^2\cos t\, dt = \pi a^2 - a^2\int_0^{2\pi}\cos t\, dt = \pi a^2}$$

$$= \frac{1}{2}\int_{2\pi}^{0} (-a^2(1+\cos^2 t - 2\cos t) + a^2\sin t\cdot t - a^2\sin^2 t)dt = \frac{1}{2}\int_{2\pi}^{0} (-a^2 - a^2 + \cancel{4}2a^2\cos t + a^2 t\sin t)dt$$

$$= \frac{1}{2}\int_{2\pi}^{0} 2a^2(\cos t - 1) + a^2 t\sin t\, dt$$

$\overset{3\pi a^2}{\shortparallel}$

$2\pi a^2 \quad + \pi a^2$

$+\int_0^{\pi}\cos t dt$.

$\cancel{t/2}[t\cos t(-1)a^2\Big|_0^{2\pi}]$

$(-1)\ t d\cos t$

~~$t\sin t\Big|_0^{2\pi}$~~ $- \frac{t}{2}$

$\frac{1}{2}\sin t\, dt^2 = \quad t^2\sin t\Big|_0^{2\pi} - \int t^2\cos t\, dt$

$= \frac{1}{2}\int_0^{2\pi}(2a^2(1-\cos t) \mp \underline{a^2 t\sin t})dt. = \frac{1}{2} 2a^2\cdot 2\pi - a^2\sin t\Big|_0^{2\pi} + $ ~~$\int_0^{2\pi} t^2\cos t\, dt$~~.

(14): $f(x,y)$ 在 $D:\{(x,y)\mid x^2+y^2\le 1\}$ 有连续二阶偏导数. 且有 $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(x^2+y^2)^2$ 则 J:

$$I=\iint_{x^2+y^2\le 1}\left(\frac{x}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial y}\right)dxdy$$

($x=r\sin\theta$ $y=r\cos\theta$)

$$=\int_0^1 r\,dr\int_0^{2\pi}\left(\cos\theta\frac{\partial f}{\partial x}+\sin\theta\frac{\partial f}{\partial y}\right)^{r}d\theta=\int_0^1 dr\int_0^{2\pi}f'_x dy-f'_y dx=\int_0^1 dr\oint_{\partial D}f'_x dy-f'_y dx$$

$$\overset{\text{Green}}{=}\int_0^1 dr\iint_D(f_{xx}+f_{yy})dxdy=\int_0^1 dr\iint_{D'}(x^2+y^2)^2dxdy=\int_0^1 dr\int_0^{2\pi}d\theta\int_0^r\rho^5d\rho$$

$$=\int_0^1 dr\int_0^{2\pi}d\theta\cdot\tfrac{1}{6}r^6=2\pi\cdot\tfrac{1}{6}\cdot\tfrac{1}{7}=\frac{\pi}{21}\ \square$$

曲线积分与路径无关: 单连通且有 $\boxed{\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}}\Leftrightarrow\boxed{\oint_L Pdx+Qdy}\Leftrightarrow\boxed{Pdx+Qdy=du}$

$\Rightarrow u(x,y)$: $du=Pdx+Qdy$. 势函数.

则: $\int_L Pdx+Qdy=\int_{A(x_1,y_1)}^{B(x_2,y_2)}Pdx+Qdy=u(x)\Big|_{A(x_1,y_1)}^{B(x_2,y_2)}=u(B)-u(A)$.

$u(x,y)=\int_{x_0}^x P(t,y_0)dt+\int_{y_0}^{y}Q(x,u)du+C$. or $u(x,y)=\int_{x_0}^x P(x,y_0)dx+\int_{y_0}^y Q(x,y)dy+C$.

$=\int_{x_0}^x P(x,y)dx+\int_{y_0}^y Q(x_0,y)dy+C$.

$\mathrm{grad}\,u=\nabla u=\underline{\vec{A}(x,y)}=P(x,y)\vec{i}+Q(x,y)\vec{j}$

有势场

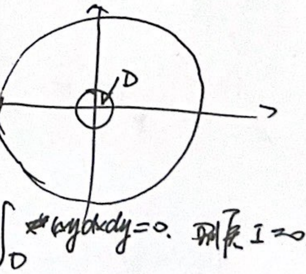
(15): $\int_C(\cos x+2xy^3)dx+(ye^y+3x^2y^2)dy$: $\Rightarrow\frac{\partial P}{\partial y}=6xy^2=\frac{\partial Q}{\partial x}\Rightarrow$ 有势场 与路径无关

$$u(x,y)=\underline{\int_0^x\cos t\,dt}+\underline{\int_0^y te^t+3x^2t^2dt}+C=\sin x+\left[te^t\Big|_0^y-\int_0^y e^tdt\right]+x^2y^3+C$$

$$=\sin x+ye^y-e^y+x^2y^3+C=\sin x+(y-1)e^y+x^2y^3+C$$

(16) $\oint_C\frac{xy(xdy-ydx)}{x^4+y^4}$ ① C 不含原点. 则 $I=0$

② . $D:\{(x,y)\mid x^4+y^4\le\delta^4\}$ 则: $\oint_C Pdx+Qdy=\oint_{C+\partial D^-}Pdx+Qdy+\oint_{\partial D}Pdx+Qdy$



$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=\frac{2xy(y^4-x^4)}{(x^4+y^4)^2}$

$=\frac{1}{\delta^4}\oint_{\partial D}-xy^2dx+x^2y\,dy=\frac{1}{\delta^4}\iint_D(4xy)dxdy=0$. 则原 $I=0$.

(17) D为R^2有界闭 $u(x,y)$在D连续: $u|_{\partial D}=0$. $\forall x\in D$. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u$.

则证: $\forall(x,y)\in D$. $u(x,y)=0$　　可以直接取出极大最大值而几

Green. $\underline{\oint_{\partial D}-u^2dx+u^2dy} = \int_D 2u\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)dxdy = \int_D 2u^2dxdy \geq 0$

$=0$　则有: $\forall x\in D$: $u\equiv 0$.　构造类似 $\int_D f(u)dudxdy$ $f(u)$恒正. 则右引向

(18). $\int_D \Delta u\, d\sigma = \oint_L \frac{\partial u}{\partial \vec{n}}ds$　$\frac{\partial u}{\partial \vec{n}}$为: $u(x,y)$沿L法向. n之方向导数:

$\vec{s}^0=(\cos\alpha,\sin\alpha) \Rightarrow \vec{n}=(\sin\alpha,-\cos\alpha)$

$\oint_L \frac{\partial u}{\partial \vec{n}}ds = \oint_L\left(\frac{\partial u}{\partial x}\sin\alpha - \frac{\partial u}{\partial y}\cos\alpha\right)ds$

Δ为Laplace算子.

$= \oint_L \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx = \int_D\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\cancel{ds}\,dxdy = \int_D \Delta u\, dxdy$

(19) ∵$u(x,y)$在 D: $x^2+y^2\leq 1$ 二阶 连续可微. $\Delta u=\cos(\pi(x^2+y^2))$

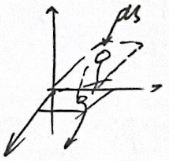
$\int_D\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)dxdy = \int_0^{2\pi}d\theta\int_0^1 r\left(r\cos\theta\frac{\partial u}{\partial x}+r\sin\theta\frac{\partial u}{\partial y}\right)\cancel{dx}\,dr$

☆ $\underline{= \int_0^1 rdr\int_0^{2\pi}\left(r\cos\theta\frac{\partial u}{\partial x}+r\sin\theta\frac{\partial u}{\partial y}\right)d\theta = \int_0^{r} rdr\int_{\partial D}(u_xdy-u_ydx)}$

$= \int_0^1 rdr.\int_{D_r^2}(u_{xx}+u_{yy})dxdy = \int_0^1 rdr\int_{D_r^2}\cos(\pi(x^2+y^2))dxdy = \int_0^1 rdr\underline{\int_0^{2\pi}d\theta}\int_0^r\cos(\rho^2\pi)\rho d\rho$

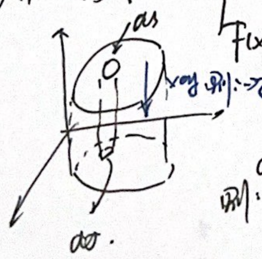
$= 2\pi\int_0^1 rdr\,\frac{1}{2\pi}\int_0^{\pi r}\cos(\rho^2\pi)d\rho^2\pi = \int_0^1 rdr\,\sin\rho^2\pi\Big|_0^r = \int_0^1 \sin(\pi r^2)rdr$

$= -\frac{1}{2\pi}\left(\overset{\cos}{\cancel{\sin}}(\pi r^2)\right)\Big|_0^1 = -\frac{(-2)}{2\pi} = \frac{1}{\pi}$!



$d\sigma = ds|\cos\gamma|$

第一曲面积分: $\begin{cases} z=z(x,y) \\ F(x,y,z)=0. \end{cases}$ 一阶有连续偏导数



xoy 则: 对其的投影

ds: 某区域 与xoy面柱的dσ. 则 法向量为 $\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$

则: $ds = \frac{d\sigma}{|\cos\gamma|} = \sqrt{1+(z_x')^2+(z_y')^2}\,d\sigma$.

$|\cos\gamma| = \frac{1}{\sqrt{(z_x')^2+(z_y')^2+1}}$　For $z=z(x,y)$. 其法向量为 $\vec{n}=\pm(z_x, z_y, -1)$ $\boxed{ds=|\vec{n}|d\sigma}$ ← 显式曲面

隐式曲面：Σ: $\begin{cases} F(x,y,z)=0 \\ F'_z \neq 0 \end{cases}$ 则: $\vec{n}=\{F'_x, F'_y, F'_z\}$. $\Rightarrow |\cos\gamma| = \frac{|F'_z|}{\sqrt{(F'_x)^2+(F'_y)^2+(F'_z)^2}}$

则: $dS = d\sigma\frac{\sqrt{(F'_x)^2+(F'_y)^2+(F'_z)^2}}{|F'_z|}$ $\Rightarrow S=\int_D \frac{1}{|F'_z|}\sqrt{(F'_x)^2+(F'_y)^2+(F'_z)^2}\,\cancel{dxdy}\,d\sigma$. D为: Σ在 xoy 上的投影区域

参数: $\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$ 则: $\vec{S_u}=\{x'_u, y'_u, z'_u\}$, $\vec{S_v}=\{x'_v, y'_v, z'_v\}$ $\Rightarrow \vec{n}=S_u\times S_v = \frac{\partial(y,z)}{\partial(u,v)}\vec{i}+\frac{\partial(z,x)}{\partial(u,v)}\vec{j}+\frac{\partial(x,y)}{\partial(u,v)}\vec{k}$

$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}$ 且有: $|\vec{n}|^2 = |S_u\times S_v|^2 = EG-F^2$. $\begin{cases} E=(x'_u)^2+(y'_u)^2+(z'_u)^2 \\ G=(x'_v)^2+(y'_v)^2+(z'_v)^2 \\ F=x'_ux'_v+y'_uy'_v+z'_uz'_v \end{cases}$

$\cos\gamma = \frac{\partial(x,y)}{\partial(u,v)}\frac{1}{\sqrt{EG-F^2}} \Rightarrow S=\int_{D_{xy}}\frac{dxdy}{|\cos\gamma|} = \int_{D_{xy}}\frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\sqrt{EG-F^2}\,dxdy = \int_{D_{uv}}\frac{1}{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\sqrt{EG-F^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$

$$\boxed{=\int_{D_{uv}}\sqrt{EG-F^2}\,dudv}\quad dS=\sqrt{EG-F^2}\,dudv.$$

$z=z(x,y) \Rightarrow I=\int_D f(x,y,z(x,y))\sqrt{1+z_x^2+z_y^2}\,dxdy$

$\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\overset{\sin}{\cancel{\cos}}\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$ 则 $\sqrt{EG-F^2}=r^2\sin\varphi$

$$\int_S f(x,y,z)\,dS = \lim_{\lambda\to0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta S_i$$

例2(2): $\int_\Sigma\frac{dS}{z} = \int_D\frac{1}{\sqrt{a^2-x^2-y^2}}\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dxdy$

Σ: $x^2+y^2+z^2=a^2$. 被 $z=h$ 截下来的.

$z=\sqrt{a^2-x^2-y^2}$. $\Rightarrow z'_x=\frac{-z}{\sqrt{a^2-x^2-y^2}}$, $z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$

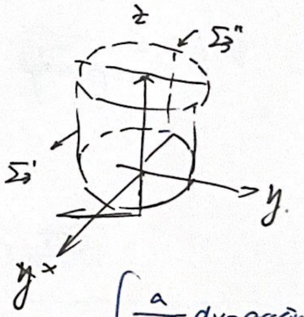
D: $x^2+y^2\le(\sqrt{a^2-h^2})^2$

$=\int_D\frac{a}{a^2-x^2-y^2}\,dxdy = \int_0^{2\pi}d\theta\int_0^{\sqrt{a^2-h^2}}\frac{ar}{a^2-r^2}\,dr$

$=\int_0^{2\pi}\left(a\ln(a^2-r^2)\cdot\frac{1}{-2}\Big|_0^{\sqrt{a^2-h^2}}\right)d\theta = \int_0^{2\pi}-\frac{a}{2}(\ln h^2-\ln a^2)\,d\theta = 2\pi\cdot\ln\frac{a}{h}.$

(2)1): $I=\int_\Sigma z\,dS$: Σ: $x^2+y^2\le a^2$

(1): $\Sigma=\cap\Sigma_1\cap\Sigma_2.\cap\Sigma_3$. Σ_1: $\begin{cases} x^2+y^2\le a^2 \\ z=0 \end{cases}$ Σ_2: $\begin{cases} x^2+y^2\le a^2 \\ z=a. \end{cases}$ Σ_3: $\begin{cases} x^2+y^2=a^2 \\ 0\le z\le a \end{cases}$



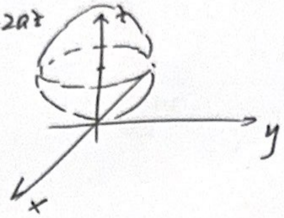
则: $I_1=\int_{\Sigma_1}z\,dS=0$. $I_2=\int_{\Sigma_2}z\,dS = a\int_{\Sigma_2}dS=\pi a^3$

$\int\frac{a}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+C$

Σ_3': $x^2+y^2=a^2$. $x=\sqrt{a^2-y^2}$. 则: $dS=(dydz)\sqrt{1+(x'_y)^2+(x'_z)^2}=\sqrt{1+0+\left(\frac{-y}{\sqrt{a^2-y^2}}\right)^2}\,dydz = \frac{a}{\sqrt{a^2-y^2}}\,dydz$

则 $I_3=2\int_\Sigma z\,dS=\int_D z\frac{a}{\sqrt{a^2-y^2}}\,dydz = a\int_0^a z\,dz\int_{-a}^{a}\frac{1}{\sqrt{a^2-y^2}}\,dy = \frac{2}{2}a^3\arcsin\frac{y}{a}\Big|_{-a}^{a}=a^3\pi$. 则: $I=a^3\pi$.

(22) $\oiint_{\Sigma k}(x^2+y^2+z^2)^3 ds$: (1): $\Sigma_1 = x^2+y^2+z^2=a^2$. (2): $\Sigma_2 : x^2+y^2+z^2=2az$



(1): $I_1 = \oint_{\Sigma k} a^2 ds = a^2\cdot 4\pi a^2 = 4\pi a^4$

$$\Rightarrow \oint_{\Sigma k}(\alpha x+\beta y+\gamma z)^2 ds = \oint_{\Sigma k}(\alpha^2x^2+\beta^2y^2+\gamma^2z^2+(2\alpha\beta xy+2\beta\gamma zy+2\alpha\gamma zx))ds = \oint_{\Sigma k}(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)ds = \frac{\alpha^2+\beta^2+\gamma^2}{3}\oint_{\Sigma k}(x^2+y^2+z^2)ds$$

$$= \frac{\alpha^2+\beta^2+\gamma^2}{3}\cdot 4\pi a^4$$

(ii): $S_1 = z_1 = a-\sqrt{a^2-x^2-y^2}$ $S_2 = a+\sqrt{a^2-x^2-y^2}$. 则: $M=(x.y.z)$. $\vec{n}=(x.y.z-a)$

$$ds = \frac{|\vec{n}|}{|z-a|}d\sigma = \underset{\alpha}{\underline{\frac{a}{\sqrt{a^2-x^2-y^2}}}}d\sigma.$$ 则: $I = \oint_{\Sigma_1+\Sigma_2} 2az\,ds = \underline{\int_{\Sigma_1} 2az\,ds} + \int_{\Sigma_2} 2az\,ds$

$$= 2a\int_{Dxy}\overset{\alpha}{a+\sqrt{a^2-x^2-y^2}}\,dxdy + 2a\int_{Dxy}\overset{\alpha}{a-\sqrt{a^2-x^2-y^2}}\,dxdy)$$

$$= 4a^3\int_{Dxy}\frac{1}{\sqrt{a^2-x^2-y^2}}dxdy = 4a^3\int_0^{2\pi}d\theta\int_0^a\frac{1}{\sqrt{a^2-r^2}}rdr$$

$$= 8\pi a^3(-\sqrt{a^2-r^2})\Big|_0^a = 8\pi a^4$$

(23): $\Sigma: \frac{x^2}{2}+\frac{y^2}{2}+z^2=1$. $(z\geq 0)$ 设 $\rho(x.y.z)$ 为: $(0,0,0)$ 到 $P(x,y,z)$ 处切平面π的距离 $\int_{\Sigma}\frac{z}{\rho(x.y.z)}ds$:

π: Σ在 $P(x.y.z)$ 处切平面:

P: 法向量 $=(x.y.2z)$: 则: $\pi: xX+yY+2zZ=2$. $\Rightarrow \rho = \frac{2}{\sqrt{x^2+y^2+4z^2}}$ 椭球面坐标: $\begin{cases} x=\sqrt{2}\sin\varphi\cos\theta \\ y=\sqrt{2}\sin\varphi\sin\theta \\ z=\cos\varphi \end{cases}$

则: $\rho = \frac{2}{\sqrt{2\sin^2\varphi+4\cos^2\varphi}}$ $\Rightarrow$. $x'_\varphi = \sqrt{2}\cos\varphi\cos\theta$. $x'_\theta = \sqrt{2}\sin\varphi(-\sin\theta)$

$y'_\varphi = \sqrt{2}\cos\varphi\sin\theta$ $y'_\theta = \sqrt{2}\sin\varphi\cos\theta$ 则: $E = 2\cos^2\varphi+\sin^2\varphi$. $F = -2\cos\varphi\sin\varphi\sin\theta\cos\theta$

$z'_\varphi = -\sin\varphi$ $z'_\theta = 0$ $G = 2\sin^2\varphi$. $+2\cos\varphi\sin\varphi\sin\theta\cos\theta$

$= 0$

$$\Rightarrow \sqrt{EG-F^2} = \sin\varphi\sqrt{4\cos^2\varphi+2\sin^2\varphi}$$

$$\boxed{\int_{\Sigma}\frac{z}{\rho(x.y.z)}ds = \cancel{\int_{D}\frac{2z\sin\varphi}{\sqrt{2\sin^2\varphi+4\cos^2\varphi}}}\cdot\sqrt{4\cos^2\varphi+2\sin^2\varphi}\,\overset{d\varphi d\theta}{\cancel{dxdy}} = \cancel{\int_D 2\cos\varphi\sin\varphi\,d\varphi d\theta} = \frac{2\pi}{2}\cdot\cancel{\int_0^{\frac{\pi}{2}} 2\cos\varphi\sin\varphi\,d\varphi} = \frac{\pi}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{2}}}$$

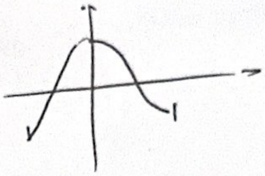
D: $x^2+y^2=2$. $z=0$

$=$

$$\iint_{\Sigma}\frac{z\,dS}{f(x,y,z)} = \iint_{D_{\varphi\theta}} \cancel{\frac{1}{2}}\cdot\frac{\sqrt{2\sin^2\varphi+\cos^2\varphi}}{2}\sqrt{2\sin^2\varphi+\cos^2\varphi}\cdot\cos\overset{\sin\varphi}{\varphi}\,d\varphi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\underline{\sin\varphi\cos\varphi}(2\cos^2\varphi+\sin^2\varphi)\,d\varphi$$

$$= \pi\underline{\int_0^{\frac{\pi}{2}}\sin 2\varphi\left(1+\frac{1+\cos 2\varphi}{2}\right)d\varphi}. = \frac{\pi}{2}\left[\int_0^{\frac{\pi}{2}}\frac{3}{2}\sin 2\varphi\,d\varphi + \underline{\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin 4\varphi}{2}\,d\varphi}\right].$$

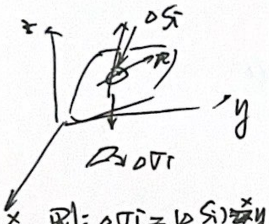
$$= \frac{\pi}{2}\left[\overset{\frac{3}{2}}{\cancel{\cos 2\varphi}}(-\cos 2\varphi)\Big|_{0}^{\frac{\pi}{2}} + 0\right]$$



$$= \cancel{-\frac{3\pi}{4}} - \frac{3\pi}{4}\cos 3\pi(\cos\pi - \overset{\cos 0)}{\cos 0\pi})$$

$$= \frac{3\pi}{2}.$$

1、第二类曲面积分: $|\pi| = \max\limits_{1\le i\le n}\{\cancel{\Delta S_i}\,\text{diag}(\Delta S_i)\} = \lambda$. $\Delta\vec{S_i}$ 在三个坐标面投影分为: $(\Delta\vec{S_i})_{yz}, (\Delta\vec{S_i})_{zx}, (\Delta\vec{S_i})_{xy}$. 则在曲面Σ可取出积分: $\lim\limits_{\lambda\to 0}\sum_{i=1}^{n} P(\xi_i,\eta_i,\zeta_i)(\Delta\vec{S_i})_{yz} + Q(\xi_i,\eta_i,\zeta_i)(\Delta\vec{S_i})_{zx} + R(\xi_i,\eta_i,\zeta_i)(\Delta\vec{S_i})_{xy}$.



$$:= \iint_{\Sigma} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy \qquad \vec{\Phi} = (P, Q, R). \quad d\vec{\sigma} = (dy\,dz, dz\,dx, dx\,dy)$$

$$:= \iint_{\Sigma}\underset{\text{第二类}}{\vec{\Phi}\cdot d\vec{\sigma}} = \iint_{\Sigma}\underset{\text{第一类}}{(\vec{\Phi}\cdot\vec{n})\,dS}.$$

则: $\Delta\sigma_i = \underline{(\Delta S_i)_{xy}} = \Delta S_i\cdot\underline{\cos\gamma}$

表示投影有向, 曲面定向: $\cos\gamma<0$ 则下侧

$\cos\gamma>0$ 则上侧:

法向量. $\underline{\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)}$.

例4: $\oiint_{\Sigma} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \cancel{3\oiint_{\Sigma} x\,dy\,dz} \oiint_{\Sigma}\left(x\frac{x}{a} + y\frac{y}{a} + z\frac{z}{a}\right)dS$ 　 Σ: $x^2+y^2+z^2=a^2$. 外侧: $\vec{n} = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)$

$$= \oiint_{\Sigma}\frac{1}{a}(x^2+y^2+z^2)\,dS = a\oiint_{\Sigma}dS = a\cdot 2\pi a^2 = 4\pi a^3$$

内侧: 则 $\vec{n'} = -\vec{n}$

例5: Σ: $(x^2+y^2)\cdot\frac{1}{2} = z$. $\underline{0\le z\le 2}$ 下侧:

则 $\iint_{\Sigma} x\,dy\,dz + y\,dz\,dx + \cancel{z}\,dx\,dy = \iint_{\Sigma}\frac{x^2+y^2+\cancel{z}1}{\sqrt{1+x^2+y^2}}\,dS$ 　 $D = \{(x,y)\mid x^2+y^2\le 4\}$

$$\vec{n} = \frac{(z_x, z_y, -1)}{\sqrt{1+(z_x)^2+(z_y)^2}} = \frac{(x, y, -1)}{\sqrt{1+x^2+y^2}}$$

$$= \iint_{D}\frac{x^2+y^2+\cancel{z}1}{\sqrt{1+x^2+y^2}}\sqrt{1+x^2+y^2}\,\cancel{dz}\,dx\,dy$$

$$\begin{cases} r\cos\theta = \cancel{y}x \\ r\sin\theta = y \end{cases}$$

$$= \iint_{D}\frac{(x^2+y^2+1)}{\cancel{2(x^2+y^2)}}\,dx\,dy$$

$$= \iint_{D'}(r^2+1)r\,dr\,d\theta = 2\pi\int_0^2(r^3+r)\,dr$$

$$= \frac{1}{2}\iint_{D}\cancel{r^3\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}d\theta\int_0^{2}r^3\,dr = \frac{1}{8}(2\pi)\frac{1}{4}(2)^4}$$

$$= \pi.$$

$$= 2\pi\left(\frac{1}{2}r^2 + \frac{1}{4}r^4\right)\Big|_0^2 = 2\pi\cdot(4+2)$$

$$= 12\pi$$

Calculation.

① $z=z(x,y)$. 则: $\iint_\Sigma R(x,y,z)dxdy = \iint_{D_{xy}} ~~R~~R(x,y,z(x,y))dxdy$. (上侧), 下侧则加 $\ominus\iint_{D_{xy}} R(x,y,z(x,y))dxdy$

② $\Sigma: \begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$ 则: $n=\left\{\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right\} \neq 0$.

$$\Rightarrow \iint_\Sigma R(x,y,z)\overset{dydz}{dxdy} = \pm\iint_{D_{yz}} R(x(u,v),y(u,v),z(u,v))\frac{\partial(z,x)}{\partial(u,v)}dzdx$$ 同理其余两个.

③、P,Q,R定义在$\Sigma: z=z(x,y)$上, 且定向为上侧则:

则: $\iint_\Sigma Pdydz+Qdzdx+Rdxdy = \iint_\Sigma (-z'_xP-z'_yQ+R)dxdy$.

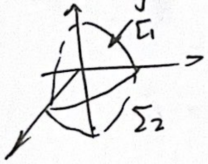
$\vec{n}=\{-z'_x,-z'_y,1\}$ 则: $dxdy=dS\cos\gamma \Rightarrow \iint_\Sigma Pdydz+Qdzdx+Rdxdy = \iint_\Sigma (P\cos\alpha+Q\cos\beta+R\cos\gamma)dS$

$$= \iint_\Sigma \frac{-z'_xP-z'_yQ+R}{\boxed{\sqrt{1+z_x'^2+z_y'^2}}☆}dS = \iint_\Sigma (-z'_xP-z'_yQ+R)\underline{\cos\gamma dS}$$

$$\underline{= \iint_\Sigma (-z'_xP-z'_yQ+R)dxdy.}$$

例: $\Sigma: x^2+y^2+z^2=a^2$. $(x,y\geq 0$, 外侧$)$: $\iint_\Sigma xyzdxdy = \iint_{\Sigma_1} xy\sqrt{a^2-x^2-y^2}dxdy + \ominus\iint_{\Sigma_2} xy\sqrt{a^2-x^2-y^2}dxdy$

下侧 (-1)* 负号.



$D: x^2+y^2\leq a^2$

$$=2\iint_{\Sigma_1} xy\sqrt{a^2-x^2-y^2}dxdy = 2\iint_{D'} ~~r^2~~ r\sin\theta\cos\theta\sqrt{a^2-r^2}\, rd\theta dr$$

$$= 2\underline{\underline{\int_0^{~~2\pi~~\ \frac{\pi}{2}} \sin 2\theta d\theta}} \underline{\int_0^a r^3\sqrt{a^2-r^2}dr}$$ $r=a\sin\varphi$. $\varphi\in(0,\frac{\pi}{2})$

$$= \underbrace{\frac{1}{2}(1-\cos 2\theta)\Big|_0^{\frac{\pi}{2}}}_{1} \underline{\int_0^{\frac{\pi}{2}} a^3\sin^3\varphi\cdot a\cos\varphi\cdot a\cos\varphi d\varphi}$$

$$= \int_0^{\frac{\pi}{2}} a^5\sin^3\varphi(1-\sin^2\varphi)d\varphi = a^5\int_0^{\frac{\pi}{2}}(\sin^3\varphi-\sin^5\varphi)d\varphi = a^5\left(\frac{2}{3}-\frac{8}{15}\right) = \frac{2}{15}a^5$$

127): $\oint_{\Sigma} x\,dydz + y\,dzdx + z\,dxdy$ $\Sigma: x^2+y^2+z^2=a^2$. 外侧:

$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$ $\varphi\in(0,\pi)$ $\theta\in(0,2\pi)$.

则: $\oint_{\Sigma} x\,dydz + y\,dzdx + z\,dxdy = a^3\int_{D\varphi\theta} (\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta + \cos^2\varphi\sin\varphi)\,d\varphi d\theta$

$\frac{\partial(x,y)}{\partial(\varphi\theta)} = \begin{vmatrix} a\cos\varphi\cos\theta & -a\sin\varphi\sin\theta \\ a\cos\varphi\sin\theta & a\sin\varphi\cos\theta \end{vmatrix} = a^2\sin\varphi\cos\varphi$.

$\frac{\partial(y,z)}{\partial(\varphi,\theta)} = \begin{vmatrix} a\cos\varphi\sin\theta & a\sin\varphi\cos\theta \\ -a\overset{\sin\varphi}{\cancel{\cos\varphi}} & 0 \end{vmatrix} = a^2\sin^2\varphi\cos\theta$

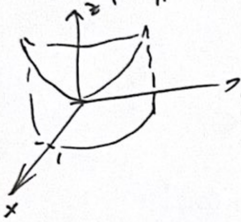
$\frac{\partial(z,x)}{\partial(\varphi,\theta)} = | a^2\sin^2\varphi\sin\theta$

$\uparrow = a^3\int_{D\varphi\theta} (\sin^3\varphi + \cos^2\varphi\sin\varphi)\,d\varphi d\theta$

$\Rightarrow = a^3\int_{D\varphi\theta} \sin\varphi\,d\varphi d\theta$

$= 2\pi a^3\int_0^{\pi} \sin\varphi\,d\varphi = 2\pi a^3(-\cos\varphi)\Big|_0^{\pi} = 2\pi a^3$

128): $\Sigma: \{(x,y,z) \mid x^2+y^2=z.\ z\in[0,1]\}$. ~~$\vec{n}$~~ $\int_{\Sigma} (2x+z)\overset{dydz}{\cancel{dxdy}} + z\,dxdy = \int_{\Sigma} (z-4x^2-2xz)\,dxdy$



法向:

$\vec{n} = (-2x, -2y, +1)$

$= \int_{\Sigma} (x^2+y^2-4x^2-2x(x^2+y^2))\,dxdy$

$dydz = (\cos\alpha)\,dS = \frac{\cos\alpha}{\cos\gamma}\,dxdy = -2x\,dxdy$ $= \int_{\Sigma xy} (-3x^2+y^2)\,dxdy$

$dxdy = (\cos\gamma)\,dS$

(坐标转换)

$D: x^2+y^2\le 1$

$= -\int_D (x^2+y^2)\,dxdy = -\int_0^{2\pi} d\theta\int_0^1 r^3\,dr = -2\pi\frac{1}{4}. = -\frac{\pi}{2}$

129): $\Sigma: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $\int_{\Sigma} \frac{1}{x}dydz + \frac{1}{y}dzdx + \frac{1}{z}dxdy = \int_{\Sigma\Sigma}$ ~~$\frac{1}{x}dydz$~~ $+$ $Dxy: \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

①. $\int_{\Sigma} \frac{1}{z}dxdy = \int_{\Sigma上} \frac{1}{z}dxdy + \int_{\Sigma下} \frac{1}{z}dxdy =$

$\begin{cases} x = a\overset{r}{\vee}\cos\theta \\ y = br\sin\theta \end{cases}$ $\theta\in[0,2\pi].\ r\in[0,1]$

$= 2\int_{Dxy} \frac{1}{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dxdy.$

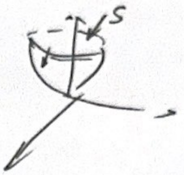
$= \frac{2abr}{c}\int_0^{2\pi} d\theta\int_0^1 \frac{r\,dr}{\sqrt{1-r^2}} = \frac{2abr}{c}2\pi\cdot(-\sqrt{1-r^2})\Big|_0^1 = \frac{4\pi ab.}{c}$

$\Rightarrow \int_{\Sigma} \frac{1}{x}dydz + \int_{\Sigma} \frac{1}{y}dzdx + \frac{1}{z}dxdy = 4\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$

Gauss: Ω由分片光滑曲面Σ围成，则有: $\underline{\oint_{\partial\Sigma} Pdydz+Qdzdx+Rdxdy = \int_\Omega (\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dxdydz}$.

第二类曲面积分与重积分.

$\boxed{\text{Σ取外侧}}$



(30): $\int_\Sigma \overset{P}{\underline{(2x+z)}}\cancel{dxdy}dydz+zdxdy$: Σ: {(x,y,z) | $x^2+y^2=z$, $z\in[0,1]$}. 上侧.

D: $x^2+y^2\le 1$

~~$=\int_\Omega 2dxdydz = 2\int_D (x^2+y^2)dxdy = 2\int_0^{2\pi}d\theta\int_0^1 r^3dr$~~ S: { $x^2+y^2\le 1$, $z=1$. 取下侧.

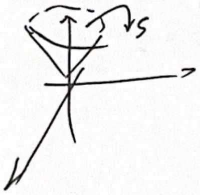
$=(\oint_{\Sigma+S}-\int_S)(2x+z)dydz+zdxdy = -\int_\Omega (2+1)dxdydz - \underline{\int_S (2x+z)dydz+zdxdy}$

$=-3\int_\Omega dxdydz - (-1)\underline{\int_S dxdy} = -\frac{3}{2}\pi+\pi = -\frac{\pi}{2}$

$\int_\Omega dxdydz = \int_D (1-(x^2+y^2))dxdy = \int_{D\theta r} r(1-r^2)drd\theta = 2\pi\int_0^{2\pi}d\theta\int_0^1(r-r^3)dr = 2\pi\cdot(\frac{1}{2}r^2-\frac{1}{4}r^4)\Big|_0^1 = \frac{2}{4}\pi = \frac{1}{2}\pi$

$\int_0^1 dz\underline{\int_{Dz}dxdy} = \int_0^1 z\pi dz = \pi/2$.

(31) Σ. $z=x^2+y^2$在 $z\in[0,h]$. 下侧:(即外侧): $\int_\Sigma x^2dydz+y^2dzdx+z^2dxdy$ S: $\sqrt{x^2+y^2}=z$. $z=h$. 上侧.



$=(\int_{\Sigma+S}-\int_S)(\underline{x^2dydz+y^2dzdx}+z^2dxdy) = \int_\Omega (2x+2y+2z)dxdydz - \underline{\int_S h^2dxdy}$

~~$2\pi h^2$~~ $\pi h^2\cdot h^2$

$=2\underline{\int_\Omega zdv} - \pi h^4$ ~~$=2\int_{D_{xy}}\sqrt{x^2+y^2}dxdydz$~~ $=\int_0^h zdz\int_{D_z}dxdy - \pi h^4$

$=2\int_0^h \pi z^2\cdot z\cancel{dxdy}dz - \pi h^4 = 2\pi\cdot\frac{1}{4}z^4\Big|_0^h - \pi h^4 = -\frac{1}{2}\pi h^4$ □

(32): Σ: $x^2+y^2+z^2=a^2$ $\boxed{\text{外侧}}$

$\partial\Omega=\Sigma$.

$\int_\Sigma xy^2dydz+yz^2dzdx+zx^2dxdy = \int_\Omega (x^2+y^2+z^2)dxdydz$

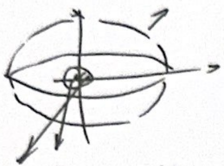
$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$.

~~$=\int_{D\varphi\theta} a^2\cdot a^2\sin\varphi$~~ $\int_{D\rho\varphi\theta}\rho^4\sin\varphi d\rho d\theta d\varphi = \int_0^a\rho^4d\rho\int_0^\pi\sin\varphi d\varphi\int_0^{2\pi}d\theta$

$=\frac{2\pi}{5}a^5\cdot(-\cos\varphi)\Big|_0^\pi = \frac{4}{5}\pi a^5$

1331: Σ: $2x^2+2y^2+z^2=6$ 外侧: $\iint_\Sigma \frac{x\,dydz+y\,dzdx+z\,dxdy}{(x^2+y^2+z^2)^{3/2}} = \iint_{\Sigma+\partial\Omega}$ (1,3).

$\frac{\partial P}{\partial x} = \frac{r^2-3x^2}{(x^2+y^2+z^2)^3} \Rightarrow \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$



Ω_1 $\partial\Omega$外侧.

Ω: $x^2+y^2+z^2\le\delta^2$ δ很小.

$= (\iint_{\Sigma+\partial\Omega_1^-} - \iint_{\partial\Omega_1^-})\oiint. = -\iint_{\partial\Omega_1^-}\frac{x\,dydz+y\,dzdx+z\,dxdy}{(x^2+y^2+z^2)^{3/2}} = \iint_{\partial\Omega_1}\frac{x\,dydz+y\,dzdx+z\,dxdy}{\delta^3}$

内侧面

$= \iint_{\partial\Omega}\frac{1}{\delta^3}(x\cos\alpha+y\cos\beta+z\cos\gamma)dS = \frac{1}{\delta^4}\iint_{\partial\Omega}(x^2+y^2+z^2)dS = \frac{1}{\delta^2}\iint_{\partial\Omega}dS =$ ④π

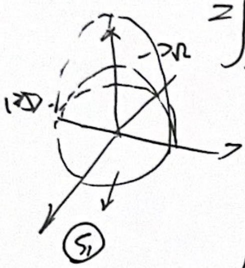
⇒

1336: Σ: $x^2+y^2+z^2=1$. $z\ge0$. $\iint_\Sigma \sin^4x\,dydz + e^{-|y|}dzdx + z^2dxdy$: $\vec{n}=$ [illegible] (x,y,z)

↓ 不能Gauss.

奇

$= \iint_\Sigma (x\sin^4x + ye^{-|y|} + z^3)dS = \iint_S z^3 dS = \iint_\Sigma z^2dxdy = \iint_{D_{xy}} z^2dxdy = \int_0^{2\pi}d\theta\int_0^1 r(1-r^2)dr = 2\pi\cdot\frac14 = \frac{\pi}{2}$



Σ: $z=2(1-x^2-y^2)$ $(z\ge0)$ 上侧. $r=\sqrt{x^2+y^2+z^2}$ $S=\begin{cases}x^2+y^2+z^2=1 & z\ge0\\ \text{下侧}\end{cases}$

$\iint_\Sigma yzr\,dydz+zxr\,dzdx+(x^2y^2-2xyr)dxdy = (\iint_{\Sigma+S}-\iint_S)\oiint$

$\frac{\partial}{\partial x}(yzr) = yz\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial}{\partial y}(zxr) = zx\frac{y}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial}{\partial z}(xyr) = xy\frac{z}{\sqrt{z^2+x^2+y^2}}$

$= \iiint_\Omega$ ~~[illegible]~~ $0\,dxdydz - [\iint_S yz\,dydz+zx\,dzdx+$ ~~[illegible]~~ $(x^2y^2-2xyr)dxdy]$

$S_1=\begin{cases}x^2+y^2=1\\ z=0\end{cases}$ 下侧

$= (\iint_{S^-+S_1}-\iint_{S_1})\oiint' = \iint_{S_1} yz\,dydz+zx\,dzdx+(x^2y^2-2xyr)dxdy$

$= \iint_{S_1^-}(x^2y^2-2xyr)dxdy = \int_0^1 dr\int_0^{2\pi}(r^5\sin^2\theta\cos^2\theta - \underbrace{2r^3\sin\theta\cos\theta}_{}\overset{0}{})d\theta$

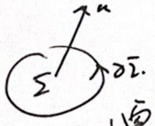
~~$= \int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta\cdot\int_0^1 r^4dr$~~ $\frac15$ $= \frac{1}{6}\int_0^{2\pi}(\sin^2\theta-\sin^4\theta)d\theta$

~~$= \frac18\int_0^{2\pi}\sin^2 2\theta\,d\theta\cdot\frac15$~~ $= \frac{4}{6}\left(\frac{\pi}{2}\left(\frac12-\frac12\times\frac34\right)\right)$

$= \frac{2\pi}{83}\cdot\frac18 = \frac{\pi}{24}$

[illegible]: Stokes公式: 光滑曲面Σ, 其边界∂Σ.

定向: ∂Σ 定向为: S和法向成右手系: ~~$\iint_\Sigma\begin{vmatrix}dydz & dzdx & dxdy\\ P & Q & R\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix} = \iint_\Sigma$~~ [illegible]



(牢记公式)

$$\iint_\Sigma\begin{vmatrix}dydz & dzdx & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix} = \iint_\Sigma \underbrace{(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dydz}_{\downarrow \text{Green.}} + (\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dzdx + (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$$

$$= \oint_{\partial\Sigma} Pdx+Qdy+Rdz$$

第二类曲线 <=> 第二类曲面积分

(36): $\oint_L y\,dx+z\,dy+x\,dz$ 　 $L: x^2+y^2+z^2=a^2$ $x+y+z=0$ 　x正向看逆时针

~~$\vec{n}=(-x,-y,-z)$~~

$\vec{n}=(-1,-1,-1)$

$$\oint_L y\,dx+z\,dy+x\,dz=\iint_\Sigma\begin{vmatrix}dxdy & dzdx & dxdy\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & z & x\end{vmatrix}=\iint_\Sigma(-1)\,dxdy+(-1)\,dzdx+(-1)\,dxdy$$

$$=-\frac{1}{3}\iint_\Sigma\left(\frac{3}{\sqrt{3}}\right)dS=-\sqrt{3}\iint_\Sigma dS=-\sqrt{3}\pi a^2$$

(37): $S: x^2+y^2+z^2=1,\ x,y,z\geq0$ 球心方向l的为逆时针.

$$I=\oint_L(y^2-z^2)\overset{dx}{\cancel{dydz}}+(z^2-x^2)\overset{dy}{\cancel{dzdx}}+(x^2-y^2)dz\cancel{dx}=\frac{1}{1}\iint_S\begin{vmatrix}-x & -y & -z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y^2-z^2 & z^2-x^2 & x^2-y^2\end{vmatrix}dS=4\underline{\iint_S(xy+y^2+z+1)dS}\ \ 24$$

画.