重积分: 与曲线曲面积分

(1): $\displaystyle\int_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,d\sigma=\frac{2}{3}\pi a^3$. (2): $\displaystyle\int_{x^2+y^2\le a^2}\sqrt{x^2+y^2}=\frac{2}{3}\pi a^3=(\pi a^2\cdot a-\frac{1}{3}\pi a^3)$. 圆柱-圆锥

公式: $\displaystyle\int_0^a\sqrt{a^2-x^2}\,dx=\frac{\pi}{4}a^2$. $\displaystyle\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\begin{cases}\cancel{(2m+1)}\dfrac{(2m-1)!!}{(2m)!!}\cdot\dfrac{\pi}{2} & n=2m\\[2mm] \dfrac{(2m)!!}{(2m+1)!!}\cdot\dfrac{\pi}{2} & n=2m+1.\end{cases}$

Wallis: $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{2n+1}}\frac{(2n)!!}{(2n-1)!!}=\sqrt{\frac{\pi}{2}}\Rightarrow\lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\frac{(2n)!!}{(2n-1)!!}=\sqrt{\pi}$

(3) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^n\sum_{j=1}^n\frac{n}{(n+i)(n^2+j^2)}=\lim_{n\to\infty}\sum_{i=1}^n\sum_{j=1}^n\frac{1}{(1+\frac{i}{n})(\frac{j^2}{n^2}+1)}\frac{1}{n^2}=\int_{[0,1]\times[0,1]}\frac{1}{(1+x)(1+y^2)}dxdy=\int_0^1\frac{1}{1+x}dx\int_0^1\frac{1}{1+y^2}dy$

$\displaystyle=(\ln(1+x)|_0^1)(\arctan y|_0^1)=\frac{\pi}{4}\ln 2.$

(4): $\displaystyle\lim_{\substack{n\to\infty\\ m\to\infty}}\sum_{i=1}^m\sum_{j=1}^n\frac{n}{(n^2+j^2)(m+i)}=\lim\sum_{i=1}^m\sum_{j=1}^n\frac{1}{(\frac{j^2}{n^2}+1)(1+\frac{i}{m})}\frac{1}{mn}=\int_{[0,1]^2}\frac{1}{(x^2+1)(y+1)}dxdy=\frac{\pi}{4}\ln 2$

(5) $f\in C(D,?)$. 计算: $\displaystyle\lim_{t\to0^+}\int_{x^2+y^2\le t^2}f(x,y)d\sigma\cdot\frac{1}{t^2}$;

积分中值定理: $\exists(\xi,\eta)\in D: x^2+y^2\le t^2$ $\displaystyle\int_{x^2+y^2\le t^2}f(x,y)d\sigma=\pi t^2\cdot f(\xi,\eta)$ 则: $\displaystyle\frac{1}{t^2}\int_{x^2+y^2\le t^2}f(x,y)d\sigma=f(\xi,\eta)$.

则: $\displaystyle\lim_{t\to0^+}\frac{1}{t^2}\int_{x^2+y^2\le t^2}f(x,y)d\sigma=\lim_{t\to0^+}f(\xi,\eta)=f(0,0)$ 则π

(6): $D: (x-1)^2+(y-1)^2\le 2$. $\displaystyle I_1=\int_D\frac{x+y}{4}d\sigma$. $\displaystyle I_2=\int_D\sqrt{\frac{x+y}{4}}\,d\sigma$. $\displaystyle I_3=\int_D\sqrt[3]{\frac{x+y}{4}}\,d\sigma$

$0\le\frac{x+y}{4}\le1$. 则 $\frac{x+y}{4}\le\sqrt{\frac{x+y}{4}}\le\sqrt[3]{\frac{x+y}{4}}\Rightarrow I_1\le I_2\le I_3$.

Thm: $f(x,y)\in R(D)$. $D=[a,b]\times[c,d]$ $\boxed{\forall x\in[a,b].\ \int_c^d f(x,y)dy \text{ 均存在.}}$ 则: $\displaystyle\int_a^b dx\int_c^d f(x,y)dy$ 存在且与: $\displaystyle\int_D f(x,y)d\sigma$ 相等.

若: $\boxed{f(x,y)\in C(D)}$. $D=[a,b]\times[c,d]$. 则: 有: $\displaystyle\int_D f(x,y)dxdy=\int_a^b dx\int_c^d f(x,y)dy=\int_c^d dy\int_a^b f(x,y)dx$

连续.

$f(x,y)\in C(D)$. $D=\{(x,y)\mid \varphi_1(x)\le y\le \varphi_2(x)\ \ \forall a\le x\le d\}$ $\varphi_1(x)$. $\varphi_2(x)$ 在 $[a,d]$ 连续,则有.

$$\iint_D f(x,y)\,dx\,dy=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$ x区域! 同理有: y区域. 记: 亦可为延拓到 $[a,b]\times[c,d]$

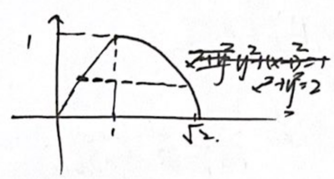
例: $f(x)\in C[a,b]$ 则: $\left(\int_a^b f(x)dx\right)^2\le (b-a)\int_a^b\{f(x)\}^2dx$

$$F(x)=(x-a)\int_a^x\{f(t)\}^2dt-\left(\int_a^x f(t)dt\right)^2\Rightarrow F(b)\ge F(a)=0$$

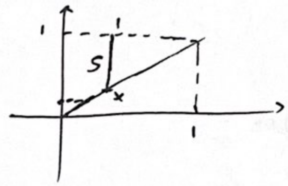
$$F'(x)=\int_a^x\{f(t)\}^2dt+(x-a)f^2(x)-2f(x)\int_a^x f(t)dt=\int_a^x\{f(t)-f(x)\}^2dt\ge 0$$

则: $F(x)\uparrow$

(8): $$\int_0^1 dx\int_0^x f(x,y)\,dx\,dy+\int_1^{\sqrt2}dx\int_0^{\sqrt{2-x^2}}f(x,y)\,dy$$



$$=\int_0^1 dy\int_y^{\sqrt{2-y^2}}f(x,y)\,dx.$$



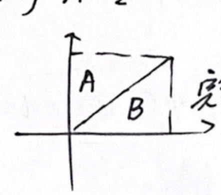
(9): $$\int_0^1 dx\int_x^1 y^2e^{-y^2}dy=\int_0^1 y^2e^{-y^2}dy\int_0^y dx=\int_0^1 y^3e^{-y^2}dy=\frac12\int_0^1 y^2e^{-y^2}dy^2=\frac12\int_0^1 ue^{-u}du$$

$$=-\frac12\int_0^1 u\,de^{-u}=-\frac12\left(ue^{-u}\Big|_0^1-\int_0^1 e^{-u}du\right)=-\frac12(u+1)e^{-u}\Big|_0^1=-\frac12(2e^{-1}-1)=\frac{1-2e^{-1}}{2}$$

(10): $f(x)\in C[0,1]$: $\int_0^1 f(x)dx=A$. 则求: $\underline{\int_0^1 dx\int_x^1 f(x)f(y)dy}=$  $\int_0^1 dy\int_0^y f(x)f(y)dx=\underline{\int_0^1 dx\int_0^x f(x)f(y)dy}$

则有: $$I=\int_0^1 dx\int_x^1 f(x)f(y)dy=\frac12\int_0^1 dx\left(\int_x^1+\int_0^x\right)f(x)f(y)dy=\frac12\int_0^1 dx\int_0^1 f(x)f(y)dy=A^2\cdot\frac12$$

另有: $F(x)=\int_0^x f(t)dt$ $F'(x)=f(x)$. $F(1)=A$



完全相对称

则有: $$\int_0^1 dx\int_x^1 (dy)f(x)f(y)=\int_x^1 f(x)dx\int_x^1 f(y)dy=\int_0^1 f(x)(F(1)-F(x))dx$$

$= \int_0^1 F(x)(F(1)-F(x))\,dF(x) = F(x)(F(1)-F(x))\Big|_0^1 - \int_0^1 F(x)\,d(F(1)-F(x)) = \int_0^1 F(x)\,dF(x) = \frac{1}{2}F^2(1) = \frac{1}{2}A^2$ · 得证

(11) $\iint_D y^2\,dx\,dy$ D: C: $\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$

D: $\{(x,y) \mid 0 \le y \le y(x),\ 0 \le x \le 2\pi a\}$ $\iint_D y^2\,dx\,dy = \int_0^{2\pi a} dx \int_0^{y(x)} y^2\,dy$

$= \frac{1}{3}\int_0^{2\pi a} y^3(x)\,dx = \frac{1}{3}\int_0^{2\pi} a^3(1-\cos t)^3\,d(a(t-\sin t))$

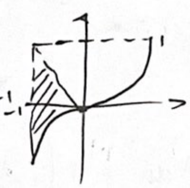
$= \frac{1}{3}a^4\int_0^{2\pi}(1-\cos t)^3(1-\cos t)\,dt$

$= \frac{1}{3}a^4\int_0^{2\pi} 16\sin^8\frac{t}{2}\,dt$

$= \frac{32a^4}{3}\int_0^{\pi}\sin^8\frac{t}{2}\,d\frac{t}{2} = \frac{32a^4}{3}\int_0^{\pi}\sin^8 u\,du = \frac{32}{3}a^4\,\frac{7!!}{8!!}\cdot\frac{\pi}{2}\cdot 2$

$= \frac{35}{12}\pi a^4$

(12): $\iint_D (3x^2\sqrt{1-y^2} + x^3\sin y)\,dx\,dy$ D: $y = x^3$, $y = 1$, $x = -1$

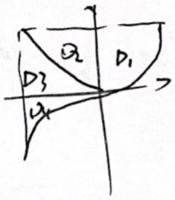


D_1: $\{(x,y) \mid x \in [-1,1],\ y \in [0,1]\}$

$= \iint_{D'} 3x^2\sqrt{1-y^2}\,dx\,dy + \iint_{D_1} x^3\sin y\,dx\,dy = 2\iint_{[0,1]\times[0,1]}(3x^2\sqrt{1-y^2} + x^2\sin y)\,dx\,dy = 2\int_0^1 dx\int_0^1 3x^2\sqrt{1-y^2}\,dy$

$\int_0^1\sqrt{1-y^2}\,dy = \int_0^{\pi/2}\cos\theta\,d\sin\theta = \int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta = \frac{\pi}{4}$ $\Rightarrow$ $= 2\int_0^1 dx^3\int_0^1\sqrt{1-y^2}\,dy = 2\cdot 1\cdot\frac{\pi}{4} = \frac{\pi}{2}$ 得证

(13): $\iint_D (\sin(x^2y) + x^2y)\,dx\,dy$ D: $y = x^3$, $y = 1$, $x = -1$.



$= \iint_{D_1+D_2} f\,dx\,dy + \iint_{D_3+D_4} f\,dx\,dy = 2\iint_{D_1} x^2y\,dx\,dy = 2\int_0^1 dx\int_{x^3}^1 x^2y\,dy = \int_0^1 (1-(x^3)^2)x^2\,dx = \int_0^1 (1-x^6)x^2\,dx = 2/9$

4

(14): $\int_D \sqrt{x^2+y^2}\, d\sigma \quad D: x^2+y^2 \le a^2. \quad \begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases} \quad D' = [0,a] \times [0, 2\pi]$

$= \int_{D'} r \cdot \left| \begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix} \right| dr d\theta$

$= \int_0^{2\pi} d\theta \int_0^a r^2 dr = \frac{1}{3} a^3 2\pi = \frac{2\pi}{3} a^3$

(15): $\int_D |x^2+y^2-1|\, d\sigma. \quad D: x^2+y^2 \le 4. \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad D': [0,2] \times [0, 2\pi]$

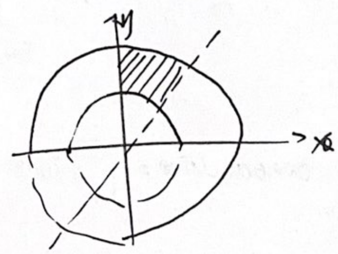
$= \int_{[0,2]\times[0,2\pi]} |r^2-1| r\, dr d\theta = 2\pi \int_0^2 |r+1||r-1| r\, d\theta = 2\pi \int_0^1 (r+1) r (1-r) dr + 2\pi \int_1^2 (r+1) r (r-1) d r$

$= 2\pi \left[\int_0^1 r - r^3 dr + \int_1^2 r^3 - r\, dr \right] = 5\pi$

(16): $\int_D \sqrt{\frac{1-(x^2+y^2)}{1+(x^2+y^2)}}\, d\sigma. \quad D: x^2+y^2 \le 1. \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases} \quad D' = [0,1] \times [0, 2\pi]$

$= \int_{D'} \sqrt{\frac{1-r^2}{1+r^2}}\, r d\theta dr \overset{u=r^2}{=} \pi \int_0^1 \sqrt{\frac{1-u}{1+u}} du = \pi \int_0^1 \frac{1-u}{\sqrt{(1+u)(1-u)}} du = \pi \int_0^1 \frac{1}{\sqrt{1-u^2}} du - \pi \int_0^1 \frac{u}{\sqrt{1-u^2}} du$

$= \pi \left(\arcsin x + \sqrt{1-u^2} \right) \Big|_0^1 = \pi \left(\frac{\pi}{2} - 1 \right)$



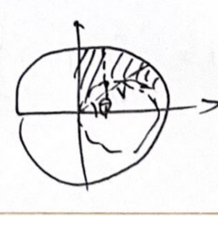
(17): $\int_D \arctan\frac{y}{x}\, d\sigma \quad \begin{cases} y = r\sin x \\ x = r\cos x \end{cases} \quad r \in [1,2]\ \theta \in [0, 2\pi]$

$\int_{D'} r \arctan(\tan\theta) dr d\theta = \int_1^2 r dr \int_0^{\pi/6} \theta d\theta = \frac{1}{2} r^2 \Big|_1^2 \cdot \frac{1}{2} \left(\frac{\pi}{6}\right)^2 = \frac{3\pi^2}{64}$

(18): $\int_0^2 dx \int_{\sqrt{2x^2-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy \quad D: \{(x,y) \mid x \in [0,2], \underbrace{\sqrt{2x-x^2} \le y \le \sqrt{4-x^2}}_{} \}$

$x^2+y^2 \le 4 \qquad x^2-2x+1+y^2 \ge 1$

$= \{(x,y) \mid x \in [0,2],\ (x-1)^2+y^2 \ge 1.\ x^2+y^2 \le 4\}$



J:

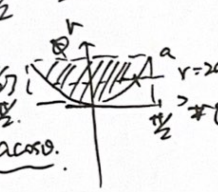
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\displaystyle\int_0^{\frac{\pi}{2}} d\theta \int_{2\cos\theta}^{2} r\cdot r\,dr = \frac{1}{3}\int_0^{\pi/2} \cancel{d\theta}\,(2^3-(2\cos\theta)^3)\,d\theta$

$\displaystyle = \frac{8}{3}\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta = \frac{8}{3}\left[\frac{\pi}{2} - \underline{\int_0^{\pi/2} \cos^2\theta\, d\sin\theta}\right] = \frac{8}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right]$

$(1-\sin^2\theta)$

$\displaystyle \sin\theta\Big|_0^{\pi/2} + \underline{\int_0^{\pi/2}\sin^2\theta\, d\sin\theta}.\ \frac{1}{3}\sin^3\theta\Big|_0^{\pi/2}$

(19): $\displaystyle\int_{-\pi/2}^{\pi/2} d\theta.\int_0^{2a\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr \quad (a>0) = \int_0^{2a} r\,dr \int_{-\arccos\frac{r}{2a}}^{\arccos\frac{r}{2a}} f(r\cos\theta, r\sin\theta)\,d\theta$

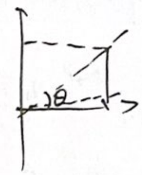


$\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

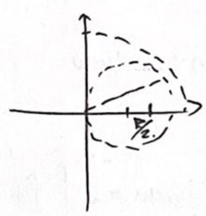
$r=2a\cos\theta \Rightarrow \theta = \arccos\frac{r}{2a}$

$0\le r\le 2a\cos\theta.$

(20). $\displaystyle\int_{[0,1]^2} f(x,y)\,d\sigma = \int_0^{\pi/4} d\theta \int_0^{\sec\theta} f(r\cos\theta, r\sin\theta)\,r\,dr.$ $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$



$\displaystyle + \int_{\pi/4}^{\pi/2} d\theta.\int_0^{\csc\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$



(21). $\displaystyle\int_D \sqrt{R^2-x^2-y^2}\,dx\,dy = 2\int_{-\pi/2}^{\pi/2} d\theta \int_0^{R\cos\theta}\sqrt{R^2-r^2}\,r\,d\theta\,dr$

$\displaystyle = \cancel{\int_{-\pi/2}^{\pi/2} d\theta \int_0^{R\cos\theta}\sqrt{R^2-r^2}\,dr^2} = 2\int_{-\pi/2}^{\pi/2} d\theta \left((\sqrt{R^2-r^2})^{3/2}\cdot\left(-\frac{2}{3}\right)\right)\Big|_0^{R\cos\theta}$

$\displaystyle = -\frac{2}{3}\int_{-\pi/2}^{\pi/2} d\theta\, \left((R^2-R^2\cos^2\theta)^{3/2}-(R^3)\right) = -\frac{2}{3}\int_{-\pi/2}^{\pi/2}(R^3-R^3(\sin\theta)^3)\,d\theta$

$\displaystyle = \frac{2}{3}R^3\int_{-\pi/2}^{\pi/2}(1-(\sin\theta)^3)\,d\theta$

$\displaystyle = +\frac{2}{3}R^3\int_{-\pi/2}^{\pi/2}\cos^3\theta\,d\theta$

$\displaystyle = \frac{4}{3}R^3\left(\frac{\pi}{2}-\int_0^{\pi}\sin^3\theta\,d\theta\right)$

$\displaystyle = \frac{2}{3}R^3\underline{\int_0^{\pi/2}\cos^3\theta\,d\theta} = \frac{4}{3}R^3\int_0^{\pi/2}(1-\sin^2\theta)\,d\sin\theta$

$\displaystyle = \frac{4}{3}R^3\left(\frac{\pi}{2}-\frac{2}{3}\right).$

$\displaystyle = \frac{4}{3}R^3\left(\sin\theta\Big|_0^{\pi/2} - \frac{1}{3}\sin^3\theta\Big|_0^{\pi/2}\right).$

6:

(22): $\int_D e^{-(x^2+y^2)} dxdy = \int_0^{\frac{\pi}{2}} d\theta \int_0^a e^{-r^2} r dr$ $D: \{(x,y) \mid x^2+y^2 \le a^2,\ x,y \ge 0\}$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \underbrace{\int_0^a e^{-r^2} dr^2} = \frac{\pi}{4} \cdot (1-e^{-r^2})\Big|_0^a = \frac{\pi}{4}(1-e^{-a^2})$

(23): $\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. (Possion).

$D: [0,a]\times[0,a]$. $D_1: x^2+y^2 \le a^2$. $D_2: x^2+y^2 \le 2a^2$ 则: $D_1 \subseteq D \subseteq D_2$.

$\underline{\int_{D_1} e^{-x^2-y^2} dxdy} \le \int_D e^{-x^2-y^2} dxdy \le \int_{D_2} e^{-x^2-y^2} dxdy$

$\int_{D_1} e^{-x^2-y^2} dxdy = \frac{\pi}{4}(1-e^{-a^2})$ $\int_{D_2} e^{-x^2-y^2} dxdy = \frac{\pi}{4}(1-e^{-2a^2})$

则: $\lim_{a\to+\infty} \int_{D_1} e^{-x^2-y^2} dxdy = \frac{\pi}{4}(1-0) = \frac{\pi}{4} = \lim_{a\to+\infty} \int_{D_2} e^{-x^2-y^2} dxdy$ 则得

且有: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{x^2}{2}} dx \overset{x=\sqrt{2}u}{=\!=} \frac{2}{\sqrt{2\pi}} \cdot \sqrt{2} \int_0^{+\infty} e^{-x^2} dx = 1$

(24): $\int_{|x|+|y|\le 1} f(x+y) dxdy = \int_{[-1,1]^2} f(u) \cdot |-\frac{1}{2}| du dv$ $\begin{cases} u = x+y \\ v = x-y \end{cases}$ 则: $u \in [-1,1]$, $v \in [-1,1]$

$= \frac{1}{2}\int_{-1}^1 dv \int_{-1}^1 f(u) du = \int_{-1}^1 f(u) du$

(25): $D.\ 2x^2+2xy+y^2 \le 1$

(即 $x^2+(x+y)^2 \le 1$.

且有 $\frac{\partial(u,v)}{\partial(x,y)} = \left|\det\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right| = 1$ $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$

则 $\begin{cases} u = x \\ v = x+y \end{cases} \Rightarrow D': u^2+v^2 = 1$

$(y = v-u)$

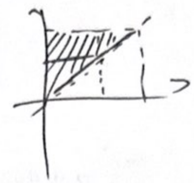
$\int_D |x| dxdy = \int_{D'} |u| du dv = \int_{-\pi}^{\pi} d\theta \int_0^1 |r\cos\theta| r dr$.

$= \int_0^{2\pi} |\cos\theta| d\theta \underline{\int_0^1 r^2 dr}$.

$= \frac{1}{3}\int_0^{2\pi} |\cos\theta| d\theta = \frac{1}{3}\cdot 4\int_0^{\frac{\pi}{2}} \cos\theta d\theta$

$= \frac{4}{3}\sin\theta\Big|_0^{\frac{\pi}{2}} = 4/3$

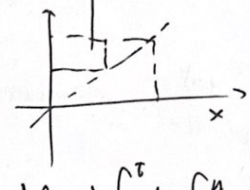
(26): $\int_0^1 dx \int_x^1 \frac{x}{\sqrt{x^2+y^2}} dy = \int_0^1 dy \int_0^{y} \frac{x}{\sqrt{x^2+y^2}} dx$



$= \int_0^1 dy \cdot (\cancel{x}\sqrt{x^2+y^2})\Big|_0^{y}$

$= \int_0^1 (\sqrt{2}y - y) dy = (\sqrt{2}-1)\int_0^1 y\, dy = \frac{\sqrt{2}-1}{2}$

(27): $\boxed{\lim_{t\to 0^+} \int_0^t dx \int_x^t \sin(xy)^2 dy \quad \le \int_0^t x^2 dx \int_x^t y^2 dy \ge \frac{1}{3}\int_0^t x^2 dx.}$



$= \lim_{t\to 0^+} \frac{1}{t^6}\int_0^t dy \int_0^y \sin(xy)^3 dx$

$(tx=u) \Rightarrow \lim_{t\to 0^+} \frac{\int_0^{t}\sin(tx^2)dx}{6t^5}$

$\frac{1}{18} = \lim_{t\to 0^+} \frac{2t\sin t^4 dt}{36t^5} = \lim_{t\to 0^+} \frac{\int_0^{t^2}\sin u^2 du}{6t^5\cdot 8}$

(28): $\int_0^1 \frac{x^3-x}{\ln x} dx = \int_0^1 \frac{x^y}{\cancel{\ln x}}\Big|_1^3 dx = \int_0^1 dx \int_1^3 \frac{x^y}{\cancel{\ln x}} dy$

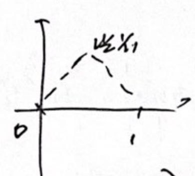
$= \int_1^3 dy \int_0^1 x^y dx = \int_1^3 dy \frac{1}{y+1}x^{y+1}\Big|_0^1$

$\ln(y+1)\Big|_1^3 = \ln 4 - \ln 2 = \ln 2$

$= \int_1^3 \frac{1}{y+1} dy = \cancel{2 + \int_1^3 y\,dy\ \frac{1}{2}y^2\Big|_1^3 = 2 + 8/2 = 4+2 = 6}$

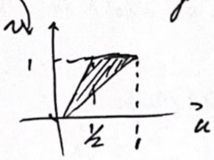
(29): $\iint_D (x+y) e^{\frac{y}{x+y}} dx dy$ $\quad D: \{(x,y) \mid 0 \le y \le x,\ x+y \le 1\}$

$\begin{cases} u = x \\ v = x+y \end{cases} \Rightarrow \begin{cases} y = -u+v \\ x = u \end{cases}$ 则 $\frac{\partial(x,y)}{\partial(u,v)} = 1.$



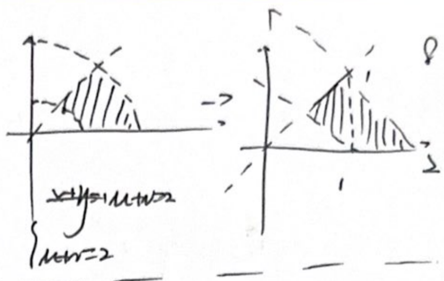
linear transformation
line → line
point → point
triangle → triangle

$\iint_{D'} v e^{u/v} du dv = \int_0^1 v dv \int_{v/2}^{v} e^{u/v} du$



$= \int_0^1 v\cdot(v e^{u/v})\Big|_{v/2}^{v} dv = \int_0^1 v^2(e-\sqrt{e})dv = \frac{1}{3}(e-\sqrt{e})$

(3): $\begin{cases} u=\sqrt{x} \\ v=\sqrt{y} \end{cases}$ 则 $\begin{cases} x=u^2 \\ y=v^2 \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & \\ & 2v \end{vmatrix} = 4uv.$



$\int_{D'} \frac{4(u+v)^4 uv}{u^4} du dv = 4\int_{D'} \frac{(u+v)^4}{u^3} v du dv$

$\begin{cases} x=u \\ \sqrt{x}+\sqrt{y}=v \end{cases}$ 则 $\frac{\partial(x,y)}{\partial(u,v)} = 2(v-\sqrt{u})$ $\quad y=(v-\sqrt{u})^2 = v^2+u-2\sqrt{u}v \le u = x$

$\Rightarrow v \le 2\sqrt{u} \Rightarrow \frac{v^2}{4} \le u \le v^2.$

则: $\int_D \frac{(\sqrt{x}+\sqrt{y})^4}{x^2} dx dy = \int_{D'} \frac{(v)^4}{u^2} 2(v-\sqrt{u}) du dv$

$= \int_1^2 dv \int_{\frac{v^2}{4}}^{v^2} \frac{v^4}{u^2}(2v-2\sqrt{u}) du dv = 2\int_1^2 v^3 dv = 15/2$

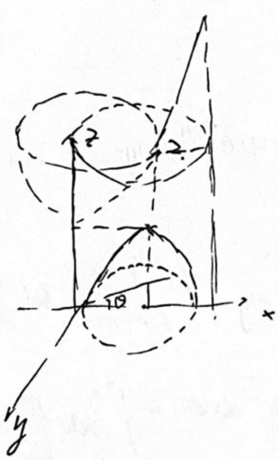
$\sigma: \sigma: \begin{cases} x=r^2\cos^4\theta \\ y=r^2\sin^4\theta \end{cases}$ 则可以推广同理: $(x)^{2/3}+(y)^{2/3}=a^{2/3} \Leftrightarrow \underline{(x^{1/3})^2+(y^{1/3})^2=a^{2/3}}$. 再得 $\begin{cases} x=a\cos^3\theta \\ y=a\sin^3\theta \end{cases}$

(3): $\int_D \frac{(x+y)\ln(1+\frac{x}{y})}{\sqrt{1-x-y}} d\sigma$. D: $(x+y\le 1.\ x,y\ge 0)$ 利用极坐标:

$0\le r\le \frac{1}{\sin\theta+\cos\theta}\quad \theta\in[0,\frac{\pi}{2}]$

则: $\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\sin\theta+\cos\theta}} \frac{(\sin\theta+\cos\theta)\ln(1+\tan\theta)}{\sqrt{1-r(\sin\theta+\cos\theta)}} \cdot r dr.$ $\quad (r(\cos\theta+\sin\theta)=u)$

$= \int_0^{\frac{\pi}{2}} \frac{\ln(1+\tan\theta)}{\underline{(\sin\theta+\cos\theta)^2}} \int_0^1 \frac{u^2}{\sqrt{1-u}} du = \frac{16}{15}\int_0^{\frac{\pi}{2}} \frac{\ln(1+\tan\theta)}{(1+\tan\theta)} d(1+\tan\theta) = 16/15$



关于x对称.

(3). $\Sigma: z=x^2+y^2+1$ 求M.合切面方程 与 Σ. $(x-1)^2+y^2=1$ 所围体积最小.

$z_0=x_0^2+y_0^2+1.$: ~~切面~~ $z-z_0=2x_0(x-x_0)+2y_0(y-y_0)$

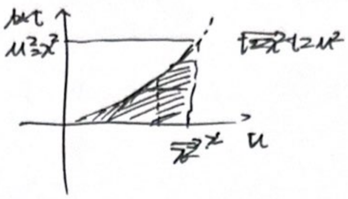
$\Rightarrow z=1-(x_0^2+y_0^2)+2x_0x+2y_0y.$

$D: \{(x,y) \mid (x-1)^2+y^2\le 1\}$ 则: $V=\int_D (1+x^2+y^2)-(1-(x_0^2+y_0^2)+2x_0x_0+\underline{2y_0y})) dx dy.$ $\quad \begin{cases} x=x_0+r\cos\theta \\ y=y_0+r\sin\theta \end{cases}$

~~$=\int_D ((x-x_0)^2+(y-y_0)^2) dx dy$~~ $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} (r^2-2x_0 r\cos\theta) r dr. = \pi((x_0-1)^2+y_0^2+1/2)/2$

$(x_0=1.\ y_0=0)$

(33): $f(x,y)$ 在 $O_1\delta(0,1)$ 可微之: $f(0,1)$. $df|_{(0,1)} = 3dx+2dy$



求: $\lim\limits_{x\to 0^+} \dfrac{\int_0^{x^2}dt\int_x^{\sqrt{t}} f(t,u)\,du}{1-\sqrt[4]{1-x^4}} = I$

$$\int_0^{x^2}dt\int_x^{\sqrt{t}} f(t,u)\,du = -\int_0^{x^2}dt\int_{\sqrt{t}}^{x} f(t,u)\,du = -\int_0^{x}du\int_0^{u^2} f(t,u)\,dt$$

则: $\lim\limits_{x\to 0^+}\dfrac{-\int_0^x du\int_0^{u^2} f(t,u)\,dt}{1-\sqrt[4]{1-x^4}} = -\lim\limits_{x\to 0^+}\dfrac{\int_0^x g(u)\,du}{1-\sqrt[4]{1-x^4}}$

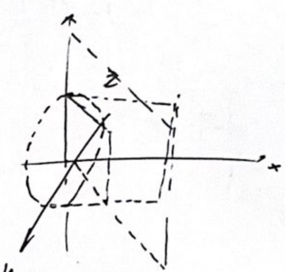
I "

$g(u) = \int_0^{u^2} f(t,u)\,dt$

$$= -\lim_{x\to 0^+}\frac{\int_0^x g(u)\,du}{\frac{1}{4}x^4} = l - \frac{g(x)}{x^3} = l - \frac{\int_0^{x^2} f(t,x)\,dt}{x^3}$$

$$= l - \frac{f(\theta x^2, x)x^2}{x^3} = -l\,\frac{f(\theta x^2, x)}{x}$$

$f(\theta x^2, x) = 0 + 3\theta x^2 + 2\theta x. \Rightarrow = -\lim\limits_{x\to 0^+}\dfrac{3\theta x^2+x\cdot 2}{x} = 2$



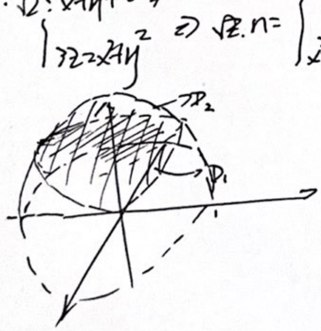
(34): $D\begin{cases} y^2+z^2=1 \\ x=0 \\ y=x \\ z=0 \end{cases} \Rightarrow \iiint_V z\,dx\,dy\,dz$

$$\Rightarrow = \int_0^1 dy\int_0^y dx\int_0^{\sqrt{1-y^2}} z\,dz = \frac{1}{2}\int_0^1 dy\int_0^y (1-y^2)\,dx = \frac{1}{2}\int_0^1 y(1-y^2)\,dy = 1/8$$

(35): Ω: $\begin{cases} z = \sqrt{x^2+y^2} \\ z=1. \end{cases}$ $\iiint_\Omega z\sqrt{x^2+y^2}\,dv.$

(i) $\iiint_\Omega z\sqrt{x^2+y^2}\,dv = \iint_D \sqrt{x^2+y^2}\,dx\,dy\int_{\sqrt{x^2+y^2}} z\,dz$ (ii): $\iiint_\Omega z\sqrt{x^2+y^2}\,dv = \int_0^1 z\,dz\iint_{x^2+y^2\le z^2}\sqrt{x^2+y^2}\,dx\,dy. \Rightarrow \int_0^1 z\,dz\int_0^{2\pi}d\theta\int_0^z r^2\,dr$

(36): Ω: $\begin{cases} x^2+y^2+z^2=4 \\ 3z = x^2+y^2 \end{cases} \Rightarrow \Omega_1\cap = \begin{cases} z=1 \\ x^2+y^2=3 \end{cases}$



$$\iiint_\Omega z\,dv = \iiint_{D_1} z\,dv + \iiint_{D_2} z\,dv = \int_0^1 z\,dz\iint_{x^2+y^2\le 3z} dx\,dy + \int_1^2 z\,dz\iint_{x^2+y^2\le 4-z^2} dx\,dy$$

$$= \int_0^1 3z^2\pi\,dz + \int_1^2 \pi z(4-z^2)\,dz = 13/4\,\pi$$

1 0.

(37): $\iiint_{\Omega}(x^2+y^2+z^2)\,dV$: $\Omega: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$D_z = \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1-\frac{z^2}{c^2}$. $S(D_z)=\pi ab(1-\frac{z^2}{c^2})$ $\iiint_{\Omega} z^2\,dV = \int_{-c}^{c} z^2\,dz \iint_{D_z} dx\,dy = \int_{-c}^{c} z^2 \pi ab(1-\frac{z^2}{c^2})\,dz = \frac{4}{15}\pi abc^2$

$\Rightarrow \Sigma\int. = \frac{4}{15}\pi abc(a^2+b^2+c^2)$

(38): $\Omega: \{(x,y,z)\mid x^2+y^2 \le 3z,\ 1\le z\le 4\}$.

$$\iiint_{\Omega}\frac{dV}{\sqrt{x^2+y^2+z^2}} = \int_1^4 dz \iint_{D_z}\frac{dx\,dy}{\sqrt{z^2+x^2+y^2}} = \int_1^4 dz \cdot \int_0^{2\pi} d\theta \int_0^{\sqrt{3z}} \frac{r\,dr}{\sqrt{r^2+z}} = 2\pi\int_1^4 \sqrt{r^2+z}\,\Big|_0^{\sqrt{3z}}\,dz$$

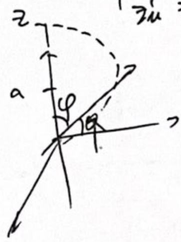
$2\pi\int_1^4 \sqrt{z}\,dz = 2\pi\cdot\frac{2}{3}\cdot z^{3/2}\Big|_1^4$

$= 2\pi\int_1^4 (2\sqrt{z}-\sqrt{z})\,dz = \cancel{\pi z^2\Big|_1^4 = 15\pi}$.

$= \frac{4}{3}\pi\cdot 7 = \frac{28}{3}\pi$.

$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix}$ $dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$

(39):



$= \boxed{\rho^2\sin\varphi}\,d\rho\,d\varphi\,d\theta$. $\begin{cases}\rho\sin\varphi\cos\theta = x\\ \rho\sin\varphi\sin\theta = y\\ \rho\cos\varphi = z.\end{cases}$ $\varphi\in[0,\pi]$.

φ: [illegible]重要

$x = r\cos\theta$. $y = r\sin\theta$ $z = z$.

$\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & +r\cos\theta\end{vmatrix} = r$.

$\rho\to\varphi\to\theta$. 积分.

(40): $(x^2+y^2+z^2)^2 = a^3 z$: Ω. $\Rightarrow$ 球面坐标: $\rho^4 = a^3\rho\cos\varphi \Leftrightarrow \rho = a\sqrt[3]{\cos\varphi}$.

$$\iiint_{\Omega} dV = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \int_0^{a\sqrt[3]{\cos\varphi}} \rho^2\sin\varphi\,d\rho$$

$$= \int_0^{2\pi} d\theta\cdot\int_0^{\pi/2} d\varphi\,(\tfrac{1}{3}\sin\varphi\,\rho^3)\Big|_0^{a\sqrt[3]{\cos\varphi}} = \int_0^{2\pi} d\theta\cdot\int_0^{\pi/2}\tfrac{1}{3}\sin\varphi\, a^3\cos\varphi\,d\varphi = \frac{2\pi}{3}a^3\int_0^{\pi/2}\tfrac{1}{2}\sin 2\varphi\,d2\varphi$$

$= \frac{\pi}{6}a^3 \frac{1-\cos 2\varphi}{\sin 2\varphi}\Big|_0^{\pi/2} = \frac{\pi a^3}{3}$